

Dynamic monopolies in directed graphs: the spread of unilateral influence in social networks

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Abstract

Irreversible dynamic monopolies were already defined and widely studied in the literature for undirected graphs. They are arising from formulation of the irreversible spread of influence such as disease, opinion, adaptation of a new product, etc., in social networks, where the influence between any two individuals is assumed to be bilateral or reciprocal. But in many phenomena, the influence in the underlying network is unilateral or one-sided. In order to study the latter models we need to introduce and study the concept of dynamic monopolies in directed graphs. Let G be a directed graph such that the in-degree of any vertex G is at least one. Let also $\tau : V(G) \rightarrow \mathbb{N}$ be an assignment of thresholds to the vertices of G . A subset M of vertices of G is called a dynamic monopoly for (G, τ) if the vertex set of G can be partitioned into $D_0 \cup \dots \cup D_t$ such that $D_0 = M$ and for any $i \geq 1$ and any $v \in D_i$, the number of edges from $D_0 \cup \dots \cup D_{i-1}$ to v is at least $\tau(v)$. One of the most applicable and widely studied threshold assignments in directed graphs is strict majority threshold assignment in which for any vertex v , $\tau(v) = \lceil (deg^{in}(v) + 1)/2 \rceil$, where $deg^{in}(v)$ stands for the in-degree of v . By a strict majority dynamic monopoly of a graph G we mean any dynamic monopoly of G with strict majority threshold assignment for the vertices of G . In this paper we first discuss some basic upper and lower bounds for the size of dynamic monopolies with general threshold assignments and then obtain some hardness complexity results concerning the smallest size of dynamic monopolies in directed graphs. Next we show that any directed graph on n vertices and with positive minimum in-degree admits a strict majority dynamic monopoly with $n/2$ vertices.

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We show that this bound is achieved by a polynomial time algorithm. This upper bound improves greatly the best known result. The final note of the paper deals with the possibility of the improvement of the latter $n/2$ bound.

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1 Introduction and motivation

The irreversible spread of influence in social networks such as spread of disease, of opinion and etc. are modeling in terms of progressive (or irreversible) dynamic monopolies in combinatorial undirected graphs [7, 13]. In this formulation the elements of the network are represented by the nodes of a graph $G = (V, E)$ and the links of the network by the edges of G . Assume that corresponding to any vertex v of G an integer value denoted by $\tau(v)$ is given. This value is called the threshold of v and the assignment $v \rightarrow \tau(v)$ is called a threshold assignment of G . Let a graph G and an assignment of thresholds τ to its vertices be given. By a τ -dynamic monopoly we mean any subset D of G such that the vertex set of G can be partitioned into subsets D_0, D_1, \dots, D_k such that $D_0 = D$ and for any $i = 1, \dots, k - 1$ each vertex v in D_{i+1} has at least $\tau(v)$ neighbors in $D_0 \cup \dots \cup D_i$. Dynamic monopolies were widely studied in the recent years with various types of threshold assignments and for various families of graphs [1, 2, 3, 5, 7, 8, 9, 10, 13]. The usual formulation of dynamic monopolies is in terms of a discrete time dynamic process defined as follows. Consider a discrete time dynamic process on the vertices of G , where some vertices of G are considered as active vertices at the beginning of the process i.e. at time zero (activeness is interpreted according to the underlying phenomenon such as disease, opinion and etc.). Denote the set of active vertices at any discrete time $t \geq 0$ by D_t . Assume that at the beginning of the process (i.e. at time zero), the vertices of a subset $D \subseteq V(G)$ are active. Hence $D_0 = D$. At each discrete time i any un-active vertex v is activated provided that v has at least $\tau(v)$ active neighbors in $D_0 \cup \dots \cup D_{i-1}$. If at the end of the process all vertices are active then the starting subset D , is called *dynamic monopoly* or simply *dynamo*. Some well-known threshold assignments for the vertices of a graph G are simple and strict majority thresholds to be defined later.

While formulating the spread of influence by undirected graphs it is assumed that the influence is a mutual property i.e. when a vertex v does influence another vertex u then u too does influence v . We notice that in some applications influence is a unilateral or one-sided relationship. For instance a person may have an influential role to another person but does not effect from the same person. For such models we have to use directed graphs and extend the concept of dynamic monopolies for

directed graphs. Throughout this paper we consider simple directed graphs. A directed graph $G = (V, E)$ is simple if it contains no loop and there exists at most one edge between any two vertices G . In particular, there exists no directed cycle of length two in G . We refer the reader for other concepts concerning directed graphs not defined in this paper to [12]. Although we consider simple directed graphs but some of our theorems are still valid for multiple directed graphs. We make a remark on this point at the concluding remarks of the paper. We present the following formal definition.

Definition 1. *Let G be a directed graph such that the in-degree of any vertex G is at least one. Let also $\tau : V(G) \rightarrow \mathbb{N}$ be an assignment of thresholds to the vertices of G such that $\tau(v) \leq \deg^{in}(v)$, for any vertex v , where $\deg^{in}(v)$ stands for the in-degree of v . A subset M of vertices of G is called a dynamic monopoly for (G, τ) if the vertex set of G can be partitioned into $D_0 \cup \dots \cup D_t$ such that $D_0 = M$ and for any $i \geq 1$ and any $v \in D_i$, the number of edges from $D_0 \cup \dots \cup D_{i-1}$ to v is at least $\tau(v)$.*

For any two vertices u and v if there is an edge from u to v then we say u is an in-neighbor of v . Let us remark that since in this model any vertex can only be effected by its in-neighbor vertices then it is assumed that all directed graphs in this paper have positive minimum in-degree. We denote the order of G by $|G|$.

Two special types of threshold assignments are mostly studies in the area of dynamic monopolies both in directed and undirected graphs. Let G be a directed graph by the simple (resp. strict) majority threshold for G we mean the threshold function τ such that $\tau(v) = \deg^{in}(v)/2$ (resp. $\tau(v) = \lceil (\deg^{in}(v) + 1)/2 \rceil$) for any vertex v of G , where $\deg^{in}(v)$ stands for the in-degree of v . By a strict majority dynamic monopoly for a graph G we mean any dynamic monopoly for G with strict majority threshold assignment. Strict majority dynamic monopolies were widely studied in the literature. First in [3], Chang and Lyuu have obtained the upper bound $23|G|/27$ for the smallest size of strict majority dynamic monopoly in any directed graph G . Then the same authors improved this bound to $0.7732|G|$ in [4]. Recently this bound improved to $2|G|/3$ in [5] and by a very shorter proof in [1] by Ackerman et al. We show in Section 2 of this paper that the smallest size of strict majority dynamic monopoly in any directed graph G is at most $|G|/2$.

The outline of the paper is as follows. In the rest of this section we discuss an upper and a lower bound for the size of dynamic monopolies with general thresholds. Then in Section 2, we obtain some hardness results concerning the complexity status of determining the smallest size of dynamic monopolies with strict majority threshold and with constant threshold assignment $\tau(v) = 2$. Next in Section 3, we first show that any strongly connected directed graph admits a strict majority dynamic monopoly with at most $\lceil |G|/2 \rceil$ vertices (Theorem 7). Then we reduce the latter bound to $\lfloor |G|/2 \rfloor$ (Theorem 8). In fact to prove this bound we need the proof of the

upper bound $\lceil |G|/2 \rceil$. Finally using this result we show that any directed graph G contains a strict majority dynamic monopoly with at most $\lfloor |G|/2 \rfloor$ vertices (Theorem 10). Such a strict majority dynamic monopoly can be obtained by a polynomial time algorithm (Remark 3). At the last section we first show that the upper bound of Theorem 10 can not be improved to any bound better than $(2/5)|G|$, i.e. to any bound with order of magnitude $(2/5)|G| - o(1)$. We end the paper with mentioning an open question about the smallest size of strict majority dynamic monopolies.

For directed graphs with general thresholds we have the following interesting result from [1]. Recall that the in-degree of any vertex v is denoted by $\deg^{in}(v)$.

Theorem 1. ([1]) *Let G be any directed graph and $\tau : V(G) \rightarrow \mathbb{N}$ any threshold assignment to the vertices of G . Then there exists a dynamic monopoly for (G, τ) with cardinality at most*

$$\sum_{v \in V(G)} \frac{\tau(v)}{\deg^{in}(v) + 1}.$$

In order to obtain a lower bound we need the following result from [10]. Let G be a graph and τ be a threshold assignment to the vertices of G . Denote the edge density of G by $\epsilon(G)$. Let also \bar{t} and t_M denote the average and maximum threshold of τ , respectively. It was proved in [10] that for any τ -dynamic monopoly M of G we have $|M| \geq |G|(1 - \frac{\epsilon(G)}{\bar{t}})(\frac{\bar{t}}{t_M})$.

By the similar proof we have the following analogous result for directed graphs.

Theorem 2. *Let G be a directed graph and τ be a threshold assignment to the vertices of G . Let also \bar{t} and t_M denote the average and maximum threshold of τ , respectively. For any τ -dynamic monopoly M of G we have*

$$|M| \geq |G|(1 - \frac{\epsilon(G)}{\bar{t}})(\frac{\bar{t}}{t_M}).$$

2 The complexity results

In this section we consider the complexity status of the smallest dynamic monopolies in directed graphs with constant threshold and strict majority threshold. The algorithmic aspects of dynamic monopolies in undirected graphs were studied by various authors (e.g. [7, 6, 2, 11]). In proving our results for directed graphs we need the results of [6] by Chen, where some hardness and inapproximability results were obtained. We only need the NP-hardness results for dynamic monopolies with strict majority and constant threshold assignments, which we present in the following uniformed format.

Theorem 3. ([6]) *To determine the smallest size of dynamic monopolies in undirected graphs is NP-hard either for strict majority threshold and for the case where all vertices have constant threshold 2.*

In the following two theorems we show that the same results hold for directed graphs.

Theorem 4. *It is an NP-complete problem to determine the smallest size of any dynamic monopoly in directed graphs where the threshold of any vertex is 2.*

Proof. By Theorem 3 it is enough to obtain a polynomial time reduction from the problem of smallest dynamic monopolies in undirected graphs with constant threshold 2 to our problem. Let G be an undirected graph whose vertices have constant threshold 2. We obtain a directed graph H from G where $\deg^{in}(x) \geq 1$ and $\tau(x) = 2$ for any $x \in H$. For any edge uv of G , we replace the edge uv by the widget $W_{u,v}$, as illustrated in the left side of Figure 1. Denote the resulting directed graph by H . We may assume that $V(G) \subseteq V(H)$. The graph H consists of $|E(G)|$ widgets $W_{u,v}$ corresponding to any edge uv of G . First note that $\deg^{in}(x) \geq 1$ for any vertex $x \in H$ and $|V(H)| = |V(G)| + 3|E(G)|$. Set $\tau(x) = 2$ for any vertex x of H . Denote the smallest size of dynamic monopolies in G and H by $d(G)$ and $d(H)$, respectively. In the following we show that $d(H) = d(G) + 2|E(G)|$. This implies the assertion of the theorem. Each of such widgets $W_{u,v}$ in H contains a vertex of type b and a vertex of type c as we labeled in the definition of $W_{u,v}$ (see $W_{u,v}$ in Figure 1). Note that all of these vertices of type b or c belong to any dynamic monopoly of H since $\tau(b) = \tau(c) = 2$ and $\deg^{in}(b) = \deg^{in}(c) = 1$. Let N be the set consisting of all vertices of type b or c in any widget of H . Since any dynamic monopoly of H contains N , the modified threshold of any vertex of type a is one. But any such vertex has an in-neighbor from the vertices of G (i.e. v). It is clear now that if M is any dynamic monopoly for G then $M \cup N$ is a dynamic monopoly for H . Conversely, let K be any dynamic monopoly for H . We have $N \subseteq K$. If a vertex of type v belongs to K we remove it from K and add the vertex v to K . The resulting set K' is still a dynamo for H whose cardinality is not greater than $|K|$. Now, $K' \setminus N$ is dynamo for G . This completes the proof. \square

The next theorem deals with the complexity of strict majority dynamic monopolies.

Theorem 5. *It is an NP-complete problem to determine the smallest size of any strict majority dynamic monopoly in directed graphs.*

Proof. By Theorem 3, it is enough to obtain a polynomial time reduction from the problem of smallest strict majority dynamic monopolies in undirected graphs to our problem. Let G be an undirected graph with strict majority threshold. We obtain a directed graph H from G where $\deg^{in}(x) \geq 1$. For any edge uv of G , we replace the

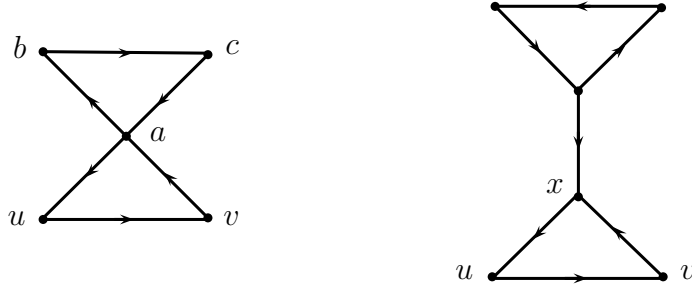


Figure 1: The widget $W_{u,v}$ (left) and the widget $W'_{u,v}$ (right)

edge uv by the widget $W'_{u,v}$, as illustrated in Fig. 1. Denote the resulting directed graph by H . We may assume that $V(G) \subseteq V(H)$. The graph H consists of $|E(G)|$ widgets $W'_{u,v}$ corresponding to any edge uv of G . Note that $\deg^{in}(x) \geq 1$ for any vertex $x \in H$ and $|V(H)| = |V(G)| + 4|E(G)|$. We consider strict majority threshold assignment for the vertices of H . Denote the smallest size of strict majority dynamic monopolies in G and H by $\text{dyn}(G)$ and $\text{dyn}(H)$, respectively. In the following we show that $\text{dyn}(H) = \text{dyn}(G) + |E(G)|$. This implies the assertion of the theorem. Each widget $W'_{u,v}$ in H consists of two directed triangles with an edge between them; an upper triangle and a lower triangle containing the directed edge uv . It is clear that any minimal dynamic monopoly in H contains exactly one vertex from any upper triangle of any widget. After activation of upper triangles the threshold of vertex x in any widget is reduced to one. So x too is activated after activation of the vertex v . Also note that $\deg_G(u) = \deg_H^{in}(u)$ for any vertex u of G . It follows that if M is any dynamic monopoly for G then $M \cup N$ is a dynamic monopoly for H where N is any set consisting of only one vertex from each widget $W'_{u,v}$ of H . Note that $|M \cup N| = |M| + |E(G)|$.

Conversely, let N be any dynamic monopoly for H . Then N intersects any upper triangle exactly once. If a vertex of type x exists in N we remove it and add the vertex v . The resulting set N' is still a dynamo whose cardinality is not greater than $|N|$. Now, $N' \cap V(G)$ is dynamo for G . This completes the proof. \square

3 The main results

We first list the terminology and notations which are used frequently in the rest of the paper. Assume that a directed graph $G = (V, E)$ is given. In the rest of the paper we denote the vertex and edge set of G by V and E , respectively.

- For any two vertices $u, v \in V$, by $uv \in E$ we mean there exists an edge from u to v

- By an in-neighbor of a vertex v we mean any vertex u such that $uv \in E$
- $N^{in}(v) = \{u \in V : uv \in E\}$
- The in-degree of v is denoted by $deg^{in}(v)$ and defined as $deg^{in}(v) = |N^{in}(v)|$
- By an ordering σ on the vertex set of G on n vertices, we mean any bijective function $\sigma : V \rightarrow \{1, 2, \dots, n\}$
- Let σ be an ordering on the vertex set of G . The function $f_\sigma : V \rightarrow \mathbb{Z}$ is defined as follows for any vertex v of G :

$$f_\sigma(v) = |N^{in}(v) \cap \{u : \sigma(u) > \sigma(v)\}| - |N^{in}(v) \cap \{u : \sigma(u) < \sigma(v)\}|$$

- By the f -value of any vertex v we mean $f_\sigma(v)$
- Let an ordering σ and $u, v \in V$ be given. We say u is appeared before v in σ (or v is appeared after u) whenever $\sigma(u) < \sigma(v)$
- Let an ordering σ and $u, v \in V$ be given such that $\sigma(u) < \sigma(v)$. We obtain a new ordering as follows

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } \sigma(x) < \sigma(u) \text{ or } \sigma(x) > \sigma(v), \\ \sigma(v) & \text{if } x = u, \\ \sigma(x) - 1 & \text{otherwise.} \end{cases}$$

We say σ' is the ordering obtained from σ by transmitting the vertex u to after v .

- Let an ordering σ and $u, v \in V$ be given such that $\sigma(u) < \sigma(v)$. We obtain a new ordering as follows

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } \sigma(x) < \sigma(u) \text{ or } \sigma(x) > \sigma(v), \\ \sigma(u) & \text{if } x = v, \\ \sigma(x) + 1 & \text{otherwise.} \end{cases}$$

We say σ' is the ordering obtained from σ by transmitting the vertex v to before u .

- Let A and B be two subsets of V . We denote the number of edges from A to B by $d(A, B)$. If $B = \{v\}$ then we simply write $d(A, v)$.

In the following we obtain an upper bound for the smallest size of dynamic monopolies in strongly connected directed graphs with strictly majority thresholds. Recall that a directed graph G is strongly connected if for any two distinct vertices u and v there exists a directed path from u to v .

Theorem 6. *Let G be a strongly connected graph.*

(i) *If G contains at least one vertex of in-degree odd, then there exists an ordering σ such that for any vertex v , $f_\sigma(v) \neq 0$.*

(ii) *If all in-degrees in G are even then there exists an ordering σ such that $f_\sigma(v) \neq 0$ for all but at most one vertex v of G .*

Proof. We prove the following stronger claim:

Claim: There exists an ordering σ satisfying the conditions of the theorem such that, for any u, v and w if $f_\sigma(u) > 0$, $f_\sigma(v) < 0$ and $f_\sigma(w) = 0$ then $\sigma(u) < \sigma(w) < \sigma(v)$.

We prove the claim by induction on $|G|$. The assertion is trivially hold when $|G| = 1$. Assume that it holds for all graphs of less than n vertices and let G be a graph with $|G| = n$. If G contains a vertex of in-degree odd then we let x be such a vertex, otherwise let x be an arbitrary vertex of G . We remove the vertex x from G and consider the strongly connected components of $G \setminus x$. Let C and C' be any two components of $G \setminus x$. We note that all the edges between C and C' are either from C to C' or from C' to C . We construct a new directed graph H as follows. Corresponding to any strongly connected component C of $G \setminus x$ we consider a vertex say v_C . Let $V(H) = \{v_C : C \text{ is a component of } G \setminus x\}$. Now for any two components C and C' we put an edge from v_C to $v_{C'}$ if and only if all the edges between C and C' in G are directed from C to C' . It is clear that H has no directed cycle. Hence the vertices of H can be ordered as $v_{C_1}, v_{C_2}, \dots, v_{C_t}$ such that if there exists an edge from v_{C_i} to v_{C_j} in H then $i < j$ (note that this kind of ordering is the same as topological sort of acyclic directed graphs). We order the strongly connected components of $G \setminus x$ according to the very ordering of $V(H)$ i.e. C_1, C_2, \dots, C_t . Note that since G is strongly connected and there exists no edge from $C_2 \cup \dots \cup C_t$ to C_1 , there exists a vertex say $u \in C_1$ such that $xu \in E$. Now since $|C_1| < |G|$, by the induction hypothesis there exists an ordering for $V(C_1)$ and its corresponding function f such that either no vertex in C_1 has f -value equal to zero or the only vertex in C_1 with zero f -value is u . Also in C_1 the vertices with positive f -value are firstly appeared, then the vertex with zero f -value and at last the vertices with negative f -value are appeared. Now we add the vertex x to the ordering of the vertices of C_1 and place it before the negative vertices and after the vertices with positive or zero f -value. We obtain an order on $C_1 \cup \{x\}$ such that the vertices with zero or positive f -value in C_1 is now positive (under the new ordering) and the negative vertices in C_1 remain

negative under the new ordering. Note that at this point we know nothing about the sign of x .

In the previous paragraph we obtained an ordering on $C_1 \cup \{x\}$ satisfying the conditions of the theorem. Assume that we have obtained a desired ordering say σ on $\{x\} \cup C_1 \cup C_2 \cup \dots \cup C_{i-1}$, where i is any value with $i \geq 2$. Again, there exists a vertex say w in C_i such that there is an edge from $\{x\} \cup C_1 \cup C_2 \cup \dots \cup C_{i-1}$ to w . By the induction hypothesis C_i admits an ordering such that the only vertex with zero value (if exists) is the very vertex w . Also the vertices with positive value are appeared first and then the vertex with zero value (if exists) and finally the vertices with negative value. Now we place the vertices of $\{x\} \cup C_1 \cup C_2 \cup \dots \cup C_{i-1}$ before the negative and after the non-negative vertices of C_i such that inside $\{x\} \cup C_1 \cup C_2 \cup \dots \cup C_{i-1}$ the ordering is the same as σ . We obtain an order say σ' for $\{x\} \cup C_1 \cup C_2 \cup \dots \cup C_{i-1} \cup C_i$. Since there is no edge from C_i to $\{x\} \cup C_1 \cup C_2 \cup \dots \cup C_{i-1}$, the positiveness and negativeness of vertices in $\{x\} \cup C_1 \cup C_2 \cup \dots \cup C_{i-1}$ does not change in σ' . Therefore all vertices in $\{x\} \cup C_1 \cup C_2 \cup \dots \cup C_{i-1} \cup C_i$ which are before x take positive f -value and all vertices after x take negative f -value. For the vertex x we have the following possibilities. If x has odd in-degree then by the definition its f -value cannot be zero. If x has an even non-zero in-degree then all other vertices have non-zero f -value and all vertices with positive value are before all vertices with negative value. The assertion holds in this case. The only other possibility is when $f_\sigma(x) = 0$. Since we have taken x arbitrarily then the assertion holds again. This completes the proof. \square

The proof of Theorem 6 shows the following remark.

Remark 1. *There exists a polynomial time recursive algorithm which outputs an ordering on the vertex set of G satisfying the conditions of Theorem 6 for any strongly connected graph G .*

Theorem 6 implies the following bound for strongly connected graphs.

Theorem 7. *Let G be a strongly connected graph on n vertices and set $\tau(v) = \lceil (deg^{in}(v)+1)/2 \rceil$ for any vertex v of G . Then the size of smallest dynamic monopoly for G is at most $\lceil n/2 \rceil$. Moreover, in case that G contains at least one vertex of odd in-degree then G contains a dynamo with no more than $n/2$ vertices.*

Proof. By Theorem 6 the vertices of G admits an ordering σ such that all but at most one vertex has f -value non-zero. Let M be the set of vertices v such that $f_\sigma(v) \geq 0$. We observe that M is a strict majority dynamo. In fact the vertices with negative f -value become active in turn according to their order in σ . Similarly if M is the set of vertices v with $f_\sigma(v) \leq 0$ then M is a dynamo. The vertices with positive f -value become active in turn according to reverse of their order in

σ . Now at least one of these sets has size at most $\lceil n/2 \rceil$. If G contains a vertex of odd in-degree then by Theorem 6 no vertex has zero f -value. Hence we obtain a dynamo with at most $n/2$ vertices. \square

In the following theorem we improve the bound obtained in Theorem 7 for strongly connected graphs.

Theorem 8. *Let G be a strongly connected graph on n vertices and set $\tau(v) = \lceil (deg^{in}(v)+1)/2 \rceil$ for any vertex v of G . Then the size of smallest dynamic monopoly for G is at most $\lfloor n/2 \rfloor$.*

Proof. Assume on the contrary that G does not admit any strict majority dynamic monopoly with $\lfloor n/2 \rfloor$ vertices. Let the ordering σ be as in Theorem 6. Then by Theorem 6, there exists at most one vertex x such that $f_\sigma(x) = 0$. Set $P = \{v : f_\sigma(v) > 0\}$ and $N = \{v : f_\sigma(v) < 0\}$. If $|P| \neq |N|$ then using the construction technique of Theorem 7 we obtain a dynamo of size at most $n/2$. If $|P| = |N|$ and $P \cup N = V$ then in this case too the same technique obtains a dynamo with at most $n/2$ vertices. Therefore by Theorem 6 the only remaining case is when $|V|$ is an odd integer, the in-degree of any vertex is even, $|P| = |N|$ and also there exists a unique vertex x such that $f_\sigma(x) = 0$. Hence by assumption for G we assume that G and σ satisfy the latter properties but G contains no dynamic monopoly with at most $\lfloor n/2 \rfloor$ vertices. We need to define sets P_1 and N_1 as follows.

$$P_1 = \{v \in P : d(P \cup \{x\}, \{v\}) < d(N, \{v\})\}$$

$$N_1 = \{v \in N : d(N \cup \{x\}, \{v\}) < d(P, \{v\})\}.$$

Note that P_1 in fact consists of the vertices v of P such that if we transmit v to after x in the ordering σ then the f -value of v corresponding to the new ordering remains positive. The similar property holds for N_1 . Let also $P_2 = P \setminus P_1$ and $N_2 = N \setminus N_1$. The arrangement of vertices in σ is P, x, N (from left to right). We rearrange the vertices in P (resp. N) in such a way that the vertices of P_2 (resp. N_1) appear firstly and then come the vertices of P_1 (resp. N_2) and the relative order of any two vertices in $P_2 \cup N_2$ is the same as in the original σ . Now the order of vertices in the rearranged σ is P_2, P_1, x, N_1, N_2 (from left to right). We denote this arrangement of σ again by σ .

In the following we first prove a sequence of facts concerning P_i and N_i , $i = 1, 2$. Then we obtain a contradiction. The contradictions to prove these facts and also the whole theorem are based on the following method. Assume that at some step of the proof, we work with an underlying order σ on the vertex set of G . If we transform σ to another order say σ' such that either the cardinality of the related set P corresponding to σ' is increased or the cardinality of the related set N corresponding to σ' is increased, then similar to the proof of Theorem 7, we obtain a dynamo with no more than $n/2$ vertices, i.e. a contraction.

Fact 1. For any $v \in P_2$ we have $d(P \cup \{x\}, v) = d(N, v)$ and for any $v \in N_2$ we have $d(N \cup \{x\}, v) = d(P, v)$.

Proof. By definition $d(P \cup \{x\}, v) \geq d(N, v)$ for any $v \in P_2$. Assume on the contrary that there exists $v \in P_2$ such that $d(P \cup \{x\}, v) > d(N, v)$. We obtain σ' from σ by transmitting v to after x . Then the size of the related set N corresponding to σ' is strictly greater than the size of previous N for σ . As we mentioned before, this implies the existence of a dynamo with at most $n/2$ vertices, i.e. is a contradiction. For the second part of the lemma, assume that there exists $v \in N_2$ with $d(N \cup \{x\}, v) > d(P, v)$. In this case we consider the ordering obtained from σ by transmitting v to before x . The rest of the proof is similar.

An interpretation of Fact 1 is that if we transmit v to after x in σ and obtain σ' then the new f -value of v becomes zero. Similarly, if we transmit v to before x then the new f -value of v becomes zero.

Fact 2. $d(P_1, x) = 0$, $d(N_1, x) = 0$.

Proof. Assume on the contrary that there exists $u \in P_1$ such that there is an edge from u to x . Let σ' be the ordering obtained from σ by transmitting u to after x . Then the size of the related P is increased strictly, a contradiction. The proof for $d(N_1, x) = 0$ is similar.

Fact 3. $d(P_1, N_2) = 0$, $d(N_1, P_2) = 0$.

Proof. Assume on the contrary that there exists an edge from some vertex $a \in P_1$ to some vertex $b \in N_2$. We obtain σ' from σ as follows. First we transmit b to before x and then transmit a to after b . We obtain $P_2, P_1 \setminus \{a\}, b, a, x, N_1, N_2 \setminus \{b\}$ (the vertices are ordered from left to right). For the resulting order by Fact 1 the corresponding set P is increased strictly with respect to the previous P . This is a contradiction. The proof for $d(N_1, P_2) = 0$ is similar.

Fact 4. For any vertex $u \in N_1$, there exists $v \in P_1$ such that $uv \in E$ and also $f_\sigma(v) = 2$.

Proof. Assume on the contrary that such a vertex does not exist. We obtain a new ordering σ' from σ by transmitting u to the beginning of the ordering. By the definition, the f -value of u becomes positive. By our hypothesis the sign of P_1 remains unchanged. Also by Fact 3, the f -value of any vertex of P_2 remains unchanged. Hence the cardinality of the corresponding set P (related to σ') is strictly increasing with respect to the previous P . This contradiction proves Fact 4.

Fact 5. For any vertex $u \in P_1$, there exists $v \in N_1$ such that $uv \in E$ and also $f_\sigma(v) = -2$.

Proof. The proof is similar to the proof of Fact 4.

Fact 6. Let u_1 and u_2 be any two vertices from N_1 . Then there does not exist $v \in P_1$ such that $u_1v, u_2v \in E$ and $f_\sigma(v) = 2$.

Proof. Assume on the contrary that such a vertex $v \in P_1$ exists. In σ we transmit v to after x and u_1 and u_2 to before x and obtain σ' . In fact the arrangement of vertices in σ' is as follows (where the vertices are ordered from left to right)

$$P_2, P_1 \setminus \{v\}, u_1, u_2, x, v, N_1 \setminus \{u_1, u_2\}, N_2$$

In σ' , the sign of f -value of any vertex of N is the same as before. The sign of $f_{\sigma'}(v)$ is negative. Therefore the size of corresponding N is strictly increased, a contradiction.

Fact 7. Let u_1 and u_2 be any two vertices from P_1 . Then there does not exist $v \in N_1$ such that $u_1v \in E$, $u_2v \in E$ and $f_\sigma(v) = -2$.

Proof. The proof is similar to the proof of Fact 6.

Fact 8. There exists a directed matching from N_1 to P_1 which saturates N_1 and also a directed matching from P_1 to N_1 which saturates P_1 .

Proof. The existence of directed matching from N_1 to P_1 is obtained by Fact 4 and Fact 6. Similarly, Fact 5 and Fact 7 imply the existence of directed matching from P_1 to N_1 .

The following is obtained from Fact 8.

Fact 9. There exists a directed matching from N_1 to P_1 which saturates $N_1 \cup P_1$ and also a directed matching from P_1 to N_1 which saturates $P_1 \cup N_1$.

We note that there is an edge from P to x , since otherwise $f(x) > 0$. By Fact 2 there is no edge from P_1 to x . Hence there is an edge from P_2 to x . Assume that $y \in P_2$ is such that $yx \in E$ and for any $y' \in P$ with $y'x \in E$, one has $\sigma(y) > \sigma(y')$. We claim that there exists a $z \in N_1 \cup N_2$ such that $yz \in E$, $f_\sigma(z) = -2$ and in σ , z is the first vertex having these properties. If such a vertex does not exist then by transmitting y to the end of σ we obtain σ' with $f_{\sigma'}(y) < 0$ and since there exists no vertex z with the above-mentioned properties, then for each $u \in N_1 \cup N_2$, $f_{\sigma'}(u) < 0$. It follows that the number of vertices with negative f -value is increased, a contradiction. Therefore the claim is proved and such a vertex z exists. For z there are two possibilities.

Case 1. $z \in N_1$.

By Fact 9, there exists $u \in P_1$ with $uz \in E$. We obtain σ' as follows.

$$P_2 \setminus \{y\} \quad P_1 \setminus \{u\} \quad z \quad u \quad x \quad y \quad N_1 \setminus \{z\} \quad N_2$$

We already had $f_\sigma(z) = -2$. We have now edges from y and u to z . Hence in σ' we have $f_{\sigma'}(z) > 0$. We already had $f_\sigma(x) = 0$ also $yx \in E$ and by Fact 2, there is no edge from z to x . Hence in σ' we have $f_{\sigma'}(x) > 0$. By the definition of P , for any

$w \in P_1 \cup P_2 \setminus \{u, y\}$ we have $f_{\sigma'}(w) > 0$. Also since $uz \in E$, there is no edge from z to u in G . By the definition of P_1 , $f_{\sigma'}(u) > 0$. In other words the number of vertices with positive f is strictly increased. This contradiction completes the proof in this case.

Case 2. $z \in N_2$ and $zx \notin E$.

In this case we construct σ' as follows.

$$P_2 \setminus \{y\} \quad P_1 \quad z \ x \ y \quad N_1 \quad N_2 \setminus \{z\}$$

By Fact 1 and that $yz \in E$ we have $f_{\sigma'}(z) = +2$. Since $yx \in E$ and $zx \notin E$ and also $f_{\sigma}(x) = 0$ we have $f_{\sigma'}(x) = +2$. This implies that the number of vertices with positive f -value is increased in σ' , a contradiction.

Case 3. $z \in N_2$ and $zx \in E$.

In this case we choose $w \in P_1 \cup P_2$ satisfying the following properties.

- (i) $f(w) = +2$
- (ii) $zw \in E$
- (iii) w is the last vertex in σ satisfying (i) and (ii).

If there does not exist such a vertex w we obtain the ordering σ' using σ by transmitting the vertex z to the beginning of σ . And then we get contradiction. If $w \in P_1$ we continue similar to case 1. If $w \in P_2$ then there are two possibilities (note that w could not be y because there is no multiple edge).

Subcase 1. $\sigma(y) < \sigma(w)$.

In this subcase, since y is the closest vertex to x in σ with $yx \in E$ then $wx \notin E$. The rest of the argument is similar to Case 2.

Subcase 2. $\sigma(w) < \sigma(y)$.

In this subcase we construct σ' using σ by transmitting the vertex z to before y . In this new ordering, the f -value of z becomes positive by Fact 1 and $yz \in E$. But we had already $f_{\sigma}(z) < 0$. The sign of vertices in P remains unchanged (i.e. positive). Now the number of vertices with positive f -value is increased in σ' . This contradiction completes the proof. \square

By Remark 1, an ordering σ satisfying the conditions of Theorem 6 can be obtained in a polynomial time. We note that any of the facts in the proof of Theorem 8 can be easily checked by a polynomial time procedure. We have therefore the following remark.

Remark 2. *There exists a polynomial time algorithm which obtains a strict majority dynamic monopoly with no more than $|G|/2$ vertices in any strongly connected graph G .*

The next theorem deals with general directed graphs. We use the fact that any directed graph can be easily decomposed into vertex disjoint strongly connected components. See [12] for basics in theory of directed graphs.

Theorem 9. *Let G be a directed graph such that no vertex of G has in-degree zero. Let C_1, C_2, \dots, C_k be the strongly connected components of G . For any vertex $v \in C_i$ define $\tau_i(v) = \lceil (deg_{C_i}^{in}(v) + 1)/2 \rceil$. Let also D_i be a dynamo for (C_i, τ_i) . Then $D_1 \cup D_2 \cup \dots \cup D_k$ is a dynamo for (G, τ) where $\tau(v) = \lceil (deg_G^{in}(v) + 1)/2 \rceil$. In case that $|C_i| = 1$ for some i , then we may choose D_i as the null set.*

Proof. Corresponding to G and C_1, \dots, C_k we construct a directed graph H as follows. Corresponding to any component C_i we consider a vertex denoted by v_i . Set $V(H) = \{v_1, v_2, \dots, v_k\}$. We put an edge from v_i to v_j if and only if there exists an edge from C_i to C_j in G . It is clear that H is acyclic. We may assume that v_1, \dots, v_k are arranged such that $v_j v_i \notin H$ whenever $i < j$. We consider the corresponding order of v_1, \dots, v_k for C_1, \dots, C_k too. Note that the in-degree of any vertex v in C_1 is the same as the in-degree of v in G . This implies that D_1 activates all vertices of C_1 in G . Assume that $D_1 \cup \dots \cup D_{i-1}$ have activated the vertices of $C_1 \cup \dots \cup C_{i-1}$. In the following we show that $C_1 \cup \dots \cup C_i$ is activated by $D_1 \cup \dots \cup D_i$. To prove this fact we need only to show that the vertices of C_i too are activated. Let v be any vertex of C_i . If there exists no edge from $V \setminus C_i$ to v then $\tau_i(v) = \tau(v)$. Otherwise the edges from $V \setminus C_i$ to v are necessarily from C_1, \dots, C_{i-1} to v . Assume that there are say, r edges from C_1, \dots, C_{i-1} to v . In this case $\tau(v) \leq \tau_i(v) + \lceil r/2 \rceil$. But the point is that after activation of all vertices in $C_1 \cup \dots \cup C_{i-1}$ including those in-neighbors of v in $C_1 \cup \dots \cup C_{i-1}$, the threshold of v practically reduces to a value no more than $\tau_i(v)$. Therefore the vertices of C_i are activated according to the activation process in (C_i, τ) using the dynamo D_i . Hence the whole graph G is activated by $D_1 \cup D_2 \cup \dots \cup D_k$.

To complete the proof let us note that when for some i , C_i consists of only one vertex, then there is no need to put C_i in the dynamic monopoly for G . Since in this case all in-neighbors of C_i are in $C_1 \cup \dots \cup C_{i-1}$. After activation of $C_1 \cup \dots \cup C_{i-1}$, C_i is also activated. This completes the proof. \square

Theorem 8 and Theorem 9 imply the following result.

Theorem 10. *Let G be a directed graph on n vertices with no vertex of in-degree zero. Set $t(v) = \lceil (deg^{in}(v) + 1)/2 \rceil$ for any vertex v of G . Then the smallest size of dynamic monopolies for G is at most $\lfloor n/2 \rfloor$.*

As we mentioned in Remark 2, in any strongly connected graph H a strict majority dynamic monopoly of size $|H|/2$ is obtained by a polynomial time algorithm. Also it is easy to decompose any directed graph into its strongly connected components. Theorem 9 and 10 imply the following.

Remark 3. *There exists a polynomial time algorithm which outputs a strict majority dynamic monopoly with cardinality at most $|G|/2$ for any directed graph G with positive minimum in-degree.*

4 Concluding remarks

In this section we first show that the upper bound of Theorem 10 can not be improved to any bound better than $(2/5)|G|$, i.e. to any bound with order of magnitude $(2/5)|G| - o(1)$. Next we present an open question concerning an upper bound for strict majority dynamic monopolies in directed graphs.

By a 2-regular directed complete graph on 5 vertices we mean an edge orientation of the complete graph on 5 vertices K_5 , where the in-degree of any vertex is 2. Since the edge set of K_5 is decomposed into two edge disjoint cycles of length 5, it is easy to obtain a 2-regular directed K_5 . In fact orient the edges of either of 5-cycles and transform each of them into a directed 5-cycle. The result is a 2-regular directed K_5 . Now consider k vertex disjoint copies of 2-regular K_5 . Add an extra vertex say x and exactly one edge from each K_5 to x so that $\deg^{in}(x) = k$. Denote the resulting graph by G_k and the smallest size of any strict majority dynamic monopoly of G_k by $\text{dyn}(G_k)$. It is obvious that any dynamo of G_k needs at least two vertices from each copy of K_5 . Therefore $\text{dyn}(G_k) \geq 2k$. In fact equality holds in the latter inequality. We have now

$$\lim_{k \rightarrow \infty} \frac{\text{dyn}(G_k)}{|G_k|} = \frac{2}{5}.$$

Based on the above fact we pose the following question.

Problem. *Is it true that any directed graph G on n vertices with positive minimum in-degree, contains a strict majority dynamic monopoly with $\lfloor 2n/5 \rfloor$ vertices?*

We note by Theorem 10 that if the above question is affirmative for strongly connected graphs then it is also affirmative for all directed graphs. We end the paper with the following remark. As we mentioned in Introduction, we considered simple directed graphs in this paper. We note that Theorem 7 still holds for multiple directed graphs. But the result of Theorem 10 is not valid for multiple directed graphs. For example consider complete directed graph $\overrightarrow{K_n}$ on n vertices, where n is odd and between any two vertices u and v there exists an edge from u to v and another edge from v to u . Observe that any strict majority dynamic monopoly for this graph needs at least $(n+1)/2$ vertices.

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